THE UNIVERSITY OF WESTERN ONTARIO
FACULTY OF ENGINEERING SCIENCE
DEPARTMENT OF ELECTRICAL ENGINEERING

E.S. 760b COMPUTATIONAL ELECTROMAGNETICS

Final Examination - April 16 - 23, 1993.

Time allotted: one week take home exam

General Instructions:

1) This is an **take home** exam.

2) There are **10** questions in total.

3) Answer **all** questions as completely as possible.

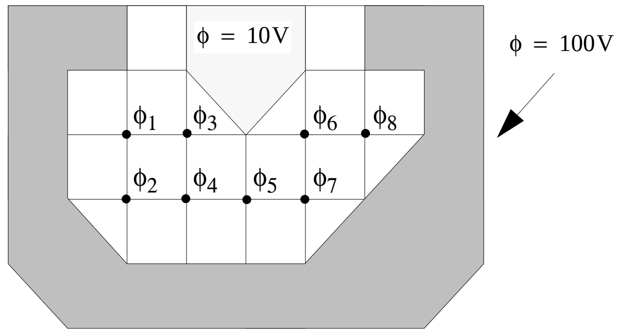
4) The marks allotted for each question are indicated in the left margin next to each question number.

5) Print clearly.

6) Clearly indicate the steps taken in your answers as part marks will be given.

Marks

10 1) For the two dimensional problem shown in the figure, determine the matrix equation which would result from applying a centered difference second order approximation to solve Laplace's equation at each grid point shown. Number you solution vector according to the numbering shown in the figure.



10 2) (a) For the matrix equation of question 1, derive the update equations of the Successive Over-relaxation iterative scheme, using an over-relaxation constant of $\omega = 1.5$, for example:

$$\phi_1^{(n+1)} = 1.5\left[\frac{1}{4}(\phi_2^{(n)} + \phi_3^{(n)} + 200) - \phi_1^{(n)}\right] + \phi_1^{(n)}.$$

(b) Use the initial guess shown in table 1 to determine the values after the first iteration as well as the relative displacement norm in the final column.

Table 1: SOR, ω = 1.5

iteration number	ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	$\frac{\lVert\Delta\phi\rVert}{\lVert\phi\rVert}$
0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	1.0	0.0
1									

15 3) The linear system of equations

$$\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}\mathbf{x} = \mathbf{b}$$

where a is real, can under certain conditions be solved by the iterative method

$$\begin{bmatrix} 1 & 0 \\ -\omega a & 1 \end{bmatrix}\mathbf{x}^{(k+1)} = \begin{bmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{bmatrix}x^{(k)} + \omega\mathbf{b}.$$

(a) For which values of a is the method convergent for $\omega = 1$.

(b) For $a = 0.5$, find the value of $\omega \in \{0.8, 0.9, 1.0, 1.1, 1.2, 1.3\}$ which minimizes the spectral radius of the matrix

$$\begin{bmatrix} 1 & 0 \\ -\omega a & 1 \end{bmatrix}^{-1}\begin{bmatrix} 1-\omega & \omega a \\ 0 & 1-\omega \end{bmatrix}$$

and explain why we would want to do this.

Marks

20 4) The loss-less transmission line equations can be written in matrix form in terms of the voltage, $V(x, t)$, and the current, $I(x, t)$, along the line as

$$\frac{\partial}{\partial t}\begin{bmatrix} V(x,t) \\ I(x,t) \end{bmatrix} + \begin{bmatrix} 0 & 1/C \\ 1/L & 0 \end{bmatrix}\frac{\partial}{\partial x}\begin{bmatrix} V(x,t) \\ I(x,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where L is the distributed inductance [H/m] and C is the distributed capacitance [F/m]. Consider the grid functions given as

$$V(i\Delta x, n\Delta t) = V_i^n \qquad I(i\Delta x, n\Delta t) = I_i^n \qquad \mathbf{u}_i^n = \begin{bmatrix} V_i^n & I_i^n \end{bmatrix}^T$$

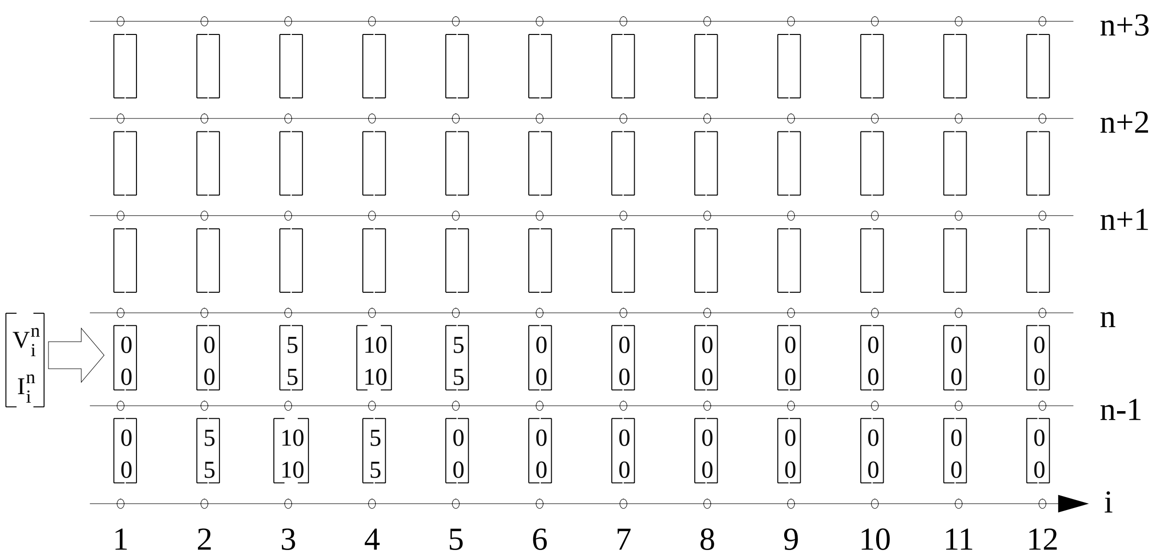
and the second order accurate centered difference approximation

$$\frac{\partial V}{\partial x} = \frac{V_{i+1}^n - V_{i-1}^n}{2\Delta x} + O(\Delta x^2)\,.$$

(a) Determine and write out the explicit Leap-Frog update equations which approximate the above coupled partial differential equations (do not use half-integer notation).

(b) Draw the computational molecule for the resulting scheme using solid dots for the voltage and hollow dots for the current.

(c) Given the value of the solution vector for the two bottom rows of the grid shown in the figure, determine the values of the grid function for the following three time steps (assume the values C = 1 F/m, L = 1 H/m, and $\Delta x = \Delta t = 1$).



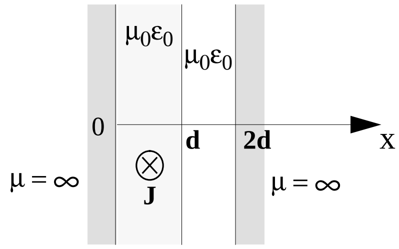
(d) Determine the number of independent grids in your scheme, and formulate a new scheme based on this one which contains only one grid (you will require half-integer notation for one of your variables).

(e) What is the stability criterion for this scheme?

10 5) Show why the one-dimensional time domain Maxwell's equations cannot be uncouple by diagonalization techniques for the case of a conducting medium ($\sigma \neq 0$).

Marks

20 6) Calculate the distribution of the magnetic flux density **B** generated by the configuration shown in the figure.



J = 2 x 10^6 [A/m^2]

d = 0.1m

(a) Solve the problem analytically using the magnetic vector potential and integrating the Poisson equation.

(b) Solve the problem using the finite element method. Discretize the 1D domain into 4 equal elements and assume a linear approximation of the solution. Compare the value of the potential and the magnetic flux density at the nodes to that of part (a).

15 7) Consider the one-dimensional Poison's equation boundary value problem given as

ODE: $\dfrac{d^2}{dx^2}u(x) = xe^x \qquad 0 \le x \le \pi$ B.C.'s: $u(0) = 0,\ u(\pi) = 0.$

(a) Show that the exact solution is given by

$$u(x) = xe^x - 2e^x + \left(\frac{2e^\pi}{\pi} - e^\pi - \frac{2}{\pi}\right)x + 2 \qquad 0 \le x \le \pi.$$

(b) Use Galerkin's method (i.e. specific instance of the Method of Moments) with basis functions given as

basis functions: $u_n = \sin(nx)\,, \qquad n = 1, 2, 3, \ldots, N$

to create an approximate expansion of the form

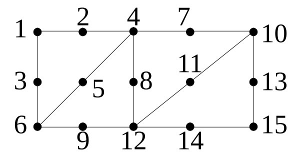
$$u(x) = \sum_{n=1}^{N} \alpha_n u_n = \underline{u}^T\underline{\alpha} = \underline{\alpha}^T\underline{u}$$

and formulate the matrix equation which approximates the solution to the above problem. Derive the components of the matrix equation for any size expansion (i.e. any N). Find the approximate solution for the case N = 3. Use the inner product defined as

inner product: $(f, g) = \int_0^\pi fg\,dx.$

(c) What do you notice about the structure of the matrix and why is it of this special form?

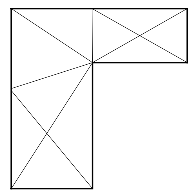
15 8) For the second order finite element mesh model, shown below,



$V_1 = V_3 = V_6 = 0$
$V_{10} = V_{13} = V_{15} = 100$

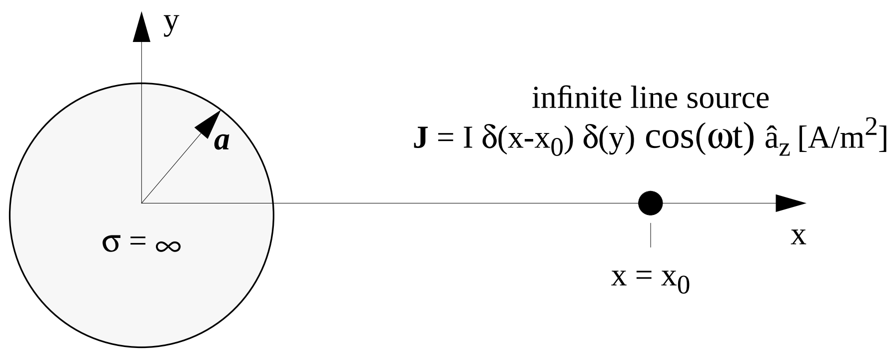
determine the stiffness matrix [S] before and after introducing the boundary conditions (assume that in each triangle there is one 90^o and two 45^o angles).

15 9) Using the ***Cuthill-McKee*** algorithm number all nodes in the following mesh.



Determine the structure of the matrix [S], the value of the bandwidth and the structure of [S] in banded form (i.e. using 0 - for zero elements and X - for nonzero elements). Choose different locations for the root node and compare values of the bandwidth.

30 10) Consider the problem of an infinite circular perfectly conducting cylinder of radius ***a*** illuminated by a line source carrying a time harmonic current I, as shown in the figure.



(a) Formulate the scattering problem via an integral formulation.

(b) Write the integral equation for the surface current density $J_z(\phi)$ on the cylinder.

(c) Write out the elements of the matrix equation obtained by using collocation and Pulse basis functions at equispaced points on the surface of the cylinder. Discuss what approximations may be used for any integrals remaining to be evaluated.